Predicate Logic - Deductive Systems

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Shin Yoo

$\mathcal{G}$ for Predicate Logic

Let's remind ourselves of semantic tableaux. Consider
$\forall x p(x) \vee \forall x q(x) \rightarrow \forall x(p(x) \vee q(x))$.

$$\neg(\forall x p(x) \vee \forall x q(x) \rightarrow \forall x(p(x) \vee q(x)))$$

$$\forall x p(x) \vee \forall x q(x), \neg\forall x(p(x) \vee q(x))$$

$\forall x p(x), \neg\forall x(p(x) \vee q(x))$	$\forall x q(x), \neg\forall x(p(x) \vee q(x))$
$\forall x p(x), \neg(p(a) \vee q(a))$	$\forall x q(x), \neg(p(a) \vee q(a))$
$\forall x p(x), \neg p(a), \neg q(a)$	$\forall x q(x), \neg p(a), \neg q(a)$
$\forall x p(x), p(a), \neg p(a), \neg q(a)$ (X)	$\forall x q(x), q(a), \neg p(a), \neg q(a)$ (X)

Upside Down, Negated...

$$
\begin{array}{cc}
\neg\forall x p(x), \neg p(a), p(a), q(a) & \neg\forall x q(x), \neg q(a), p(a), q(a) \\
| & | \\
\neg\forall x p(x), p(a), q(a) & \neg\forall x q(x), p(a), q(a) \\
| & | \\
\neg\forall x p(x), p(a) \vee q(a) & \neg\forall x q(x), p(a) \vee q(a) \\
| & | \\
\neg\forall x p(x), \forall x(p(x) \vee q(x)) & \neg\forall x q(x), \forall x(p(x) \vee q(x))
\end{array}
$$

$$
\begin{array}{c}
\neg(\forall x p(x) \vee \forall x q(x)), \forall x(p(x) \vee q(x)) \\
| \\
\forall x p(x) \vee \forall x q(x) \rightarrow \forall x(p(x) \vee q(x))
\end{array}
$$

$\mathcal{G}$ for Predicate Logic

Definition 1 (8.1)

The Gentzen system, $\mathcal{G}$, for predicate logic is a deductive system. Its axioms are sets of formulas, U, containing a complementary pair of literals. The rules of inference are the rules given for $\alpha-$ and $\beta-$formulas discussed for Proposition Logic, together with the following rules for $\gamma-$ and $\delta-$formulas.

γ	$\gamma(a)$	δ	$\delta(a)$
$\exists x A(x)$	$A(a)$	$\forall x A(x)$	$A(a)$
$\neg\forall x A(x)$	$\neg A(a)$	$\neg\exists x A(x)$	$\neg A(a)$
$\dfrac{U \cup \{\gamma, \gamma(a)\}}{U \cup \{\gamma\}}$		$\dfrac{U \cup \{\delta(a)\}}{U \cup \{\delta\}}$	

The $\delta-$rule can be applied only if the constant a does not occur in any formula in U.

$\mathcal{G}$ for Predicate Logic

γ	$\gamma(a)$	δ	$\delta(a)$
$\exists x A(x)$	$A(a)$	$\forall x A(x)$	$A(a)$
$\neg\forall x A(x)$	$\neg A(a)$	$\neg\exists x A(x)$	$\neg A(a)$
$\dfrac{U \cup \{\gamma, \gamma(a)\}}{U \cup \{\gamma\}}$		$\dfrac{U \cup \{\delta(a)\}}{U \cup \{\delta\}}$	

- $\gamma-$rule: if an existential formula and some instantiation of it are true, then the instantiation is redundant.
- $\delta-$rule: Let a be an arbitrary constand. Suppose $A(a)$ can be proved. Since a was arbitrary, the proof holds for $\forall x A(x)$. For this to work, it is essential that a is an arbitrary constant, not constrained by any other subformula.

$\mathcal{G}$: Example

Prove $\exists x \forall y p(x, y) \rightarrow \forall y \exists x p(x, y)$.

1.	$\underline{\neg\forall y p(a, y), \neg p(a, b)}, \exists x p(x, b), p(a, b)$	Axiom
2.	$\neg\forall y p(a, y), \underline{\exists x p(x, b), p(a, b)}$	γ, 1
3.	$\neg\forall y p(a, y), \underline{\exists x p(x, b)}$	γ, 2
4.	$\underline{\neg\forall y p(a, y)}, \forall y \exists x p(x, y)$	δ, 3
5.	$\underline{\neg\exists x \forall y p(x, y), \forall y \exists x p(x, y)}$	δ, 4
6.	$\exists x \forall y p(x, y) \rightarrow \forall y \exists x p(x, y)$	α, 5

Definition 2 (8.4)

The axioms of $\mathcal{H}$ for predicate logic are:

- **Axiom 1** $\vdash (A \to (B \to A))$
- **Axiom 2** $\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$
- **Axiom 3** $\vdash (\neg B \to \neg A) \to (A \to B)$
- **Axiom 4** $\vdash \forall x A(x) \to A(a)$
- **Axiom 5** $\vdash \forall x(A \to B(x)) \to (A \to \forall x B(x))$

The rules of inference are **modus ponens** and **generalisation**:

MP: $\dfrac{\vdash A \to B \quad \vdash A}{\vdash B}$, Gen.: $\dfrac{\vdash A(a)}{\vdash \forall x A(x)}$

Note that Axiom 1, 2, 3 and MP rule are generalized to any formulas in predicate logic: hence we can use any derived rules and theorems that we proved for propositional logic.

Specialisation and Generalisation

Axiom 4 can be used as an inference rule (**specialisation**):

$$\frac{U \vdash \forall x A(x)}{U \vdash A(a)}$$

Any occurrence of $\forall x A(x)$ can be replaced by $A(a)$ for any a. If $A(x)$ is true whatever the assignment of a domain element of an interpretation $\mathscr{I}$ to x, then $A(a)$ is true for the domain element that $\mathscr{I}$ assigns to a. Note that this rule holds when we have U (i.e., other assumptions).

Specialisation and Generalisation

Generalisation rule is given without a set of assumptions, U:

$$\frac{\vdash A(a)}{\vdash \forall x A(x)}$$

Suppose we allow applying generalisation to $A(a) \vdash A(a)$, to obtain $A(a) \vdash \forall x A(x)$. Consider the following interpretation for this formula: $(\mathcal{Z}, \{even(x)\}, \{2\})$. $A(a)$ is true, but obviously $\forall x A(x)$ is not true.

To cater for assumptions involved in proofs, **generalisation** can be also presented like the following:

$$\frac{U \vdash A(a)}{U \vdash \forall x A(x)}$$

provided that a does not appear anywyere in U.

Deduction Rule

$$\frac{U \cup \{A\} \vdash B}{U \vdash A \to B}$$

Theorem 1 (8.10)

The deduction rule is a sound derived rule.

Proof.

See the proof for Theorem 3.14 for propositional logic. We use the induction on the length of the proof of $U \cup \{A\} \vdash B$, and show that we can obtain a proof for $U \vdash A \to B$ without using the deduction rule. $\square$

Deduction Rule

Prof. Cont.

For $n = 1$, B is proved in a single step, which means either $B \in U \cup \{A\}$, an axiom, or a theorem. Refer to the proof of Theorem 3.14 for axioms.

If $n > 1$, the last step in the proof of $U \cup \{A\} \vdash B$ is either an one-step inference of B, an inference based on MP, or an inference based on generalisation. We focus on generalisation: if the last rule applied was generalisation, we can assume that the preceding line was $U \cup \{A\} \vdash B(a)$ (by definition, a does not appear in U or A:

$$\begin{array}{rll} i & U \cup \{A\} \vdash B(a) & \\ i+1 & U \cup \{A\} \vdash \forall x B(x) & \text{Generalisation} \end{array}$$

The inductive hypothesis is that the deduction rule holds up to line i.

$$\begin{array}{rll} i & U \cup \{A\} \vdash B(a) & \\ i' & U \vdash A \rightarrow B(a) & \text{Inductive Hypothesis, } i \\ i'+1 & U \vdash \forall x(A \rightarrow B(x)) & \text{Generalisation, } i' (a \notin U) \\ i'+2 & U \vdash \forall x(A \rightarrow B(x)) \rightarrow (A \rightarrow \forall x B(x)) & \text{Axiom 5 } (a \notin A) \\ i'+3 & U \vdash A \rightarrow \forall x B(x) & \text{MP, } i'+1, i'+2 \end{array}$$

Equivalence between $\mathcal{G}$ and $\mathcal{H}$

We already know that any proof in $\mathcal{G}$ can be mechanically converted into a proof in $\mathcal{H}$ for propositional logic. We only need to extend the existing proof to cover $\gamma-$ and $\delta-$rules.

Theorem 2 (8.11)

The rule for a $\gamma-$formula can be simulated in $\mathcal{H}$.

Proof.

Suppose we use the following $\gamma-$rule: $\dfrac{U \vee \neg\forall x A(x) \vee \neg A(a)}{U \vee \neg\forall x A(x)}$. Then:

1.	$\vdash \forall x A(x) \rightarrow A(a)$	Axiom 4
2.	$\vdash \neg\forall(x)A(x) \vee A(a)$	Propositional Deduction
3.	$\vdash U \vee \neg\forall(x)A(x) \vee A(a)$	Propositional Deduction (Weakening)
4.	$\vdash U \vee \neg\forall(x)A(x) \vee \neg A(a)$	Assumption
5.	$\vdash U \vee \neg\forall(x)A(x)$	Propositional Deduction 3, 4

$\square$

Equivalence between $\mathcal{G}$ and $\mathcal{H}$

Theorem 3 (8.12)

The rule for a $\delta-$formula can be simulated in $\mathcal{H}$.

Proof.

Suppose we use the following $\gamma-$rule: $\dfrac{U \vee A(a)}{U \vee \forall x A(x)}$. Then:

1.	$\vdash U \vee A(a)$	Assumption
2.	$\vdash \neg U \rightarrow A(a)$	Propositional Deduction, 1
3.	$\vdash \forall x(\neg U \rightarrow A(x))$	Gen., 2
4.	$\vdash \forall x(\neg U \rightarrow A(x)) \rightarrow (\neg U \rightarrow \forall x A(x))$	Axiom 5
5.	$\vdash \neg U \rightarrow \forall x A(x)$	MP, 3, 4
6.	$\vdash U \vee \forall x A(x)$	Propositional Deduction, 5

$\square$

Equivalence between $\mathcal{G}$ and $\mathcal{H}$

This is one-direction: how do we prove that any proof in $\mathcal{H}$ can be done in $\mathcal{G}$?

Examples of proofs in $\mathcal{H}$

Theorem 4 (8.14)

$\vdash A(a) \to \exists x A(x)$

Proof.

1.	$\vdash \forall x \neg A(x) \to \neg A(a)$	Axiom 4
2.	$\vdash A(a) \to \neg \forall x \neg A(x)$	Prop. (Contrapositive) 1
3.	$\vdash A(a) \to \exists x A(x)$	Duality, 2

$\square$

Examples of proofs in $\mathcal{H}$

Theorem 5 (8.15)

$\vdash \forall x A(x) \rightarrow \exists x A(x)$

Proof.

1.	$\{\forall x A(x)\} \vdash \forall x A(x)$	Assumption
2.	$\{\forall x A(x)\} \vdash \forall x A(x) \rightarrow A(a)$	Axiom 4
3.	$\{\forall x A(x)\} \vdash A(a)$	MP, 1, 2
4.	$\{\forall x A(x)\} \vdash A(a) \rightarrow \exists x A(x)$	Theorem 8.14
5.	$\{\forall x A(x)\} \vdash \exists x A(x)$	MP, 3, 4
6.	$\vdash \forall x A(x) \rightarrow \exists x A(x)$	Deduction

$\square$

Examples of proofs in $\mathcal{H}$

Theorem 6 (8.19)

- $\vdash \forall x(A \rightarrow B(x)) \leftrightarrow (A \rightarrow \forall x B(x))$
- $\vdash \exists x(A \rightarrow B(x)) \leftrightarrow (A \rightarrow \exists x B(x))$

Examples of proofs in $\mathcal{H}$

Theorem 7 (8.20)

$\vdash \forall x A(x) \leftrightarrow \forall y A(y)$

Proof.

1.	$\vdash \forall x A(x) \rightarrow A(a)$	Axiom 4
2.	$\vdash \forall y (\forall x A(x) \rightarrow A(y))$	Generalisation, 1
3.	$\vdash \forall y (\forall x A(x) \rightarrow A(y)) \rightarrow (\forall x A(x) \rightarrow \forall y A(y))$	Axiom 5
4.	$\vdash \forall x A(x) \rightarrow \forall y A(y)$	MP 2, 3

Repeat in the reverse direction. $\square$

Examples of proofs in $\mathcal{H}$

This may appear a bit counter-intuitive at first (careful with the parentheses).

Theorem 8 (8.21)

$\vdash \forall x(A(x) \rightarrow B) \leftrightarrow (\exists x A(x) \rightarrow B)$

Proof.

$\rightarrow$

1.	$\{\forall x(A(x) \rightarrow B)\} \vdash \forall x(A(x) \rightarrow B)$	Assumption
2.	$\{\forall x(A(x) \rightarrow B)\} \vdash \neg B \rightarrow \neg\forall x(A(x))$	
3.	$\{\forall x(A(x) \rightarrow B)\} \vdash \neg B \rightarrow \forall x \neg A(x)$	Axiom 5 + MP
4.	$\{\forall x(A(x) \rightarrow B)\} \vdash \neg\forall x \neg A(x) \rightarrow B$	Prop. (Contrapositive)
5.	$\{\forall x(A(x) \rightarrow B)\} \vdash \exists x A(x) \rightarrow B$	Duality
6.	$\vdash \forall x(A(x) \rightarrow B) \rightarrow \exists x A(x) \rightarrow B$	Deduction

$\square$

Examples of proofs in $\mathcal{H}$

This may appear a bit counter-intuitive at first (careful with the parentheses).

Theorem 9 (8.21)

$\vdash \forall x(A(x) \rightarrow B) \leftrightarrow (\exists x A(x) \rightarrow B)$

Proof.

$\leftarrow$

1.	$\{\exists x A(x) \rightarrow B\} \vdash \exists x A(x) \rightarrow B$	Assumption
2.	$\{\exists x A(x) \rightarrow B\} \vdash \neg\forall x \neg A(x) \rightarrow B$	Duality
3.	$\{\exists x A(x) \rightarrow B\} \vdash \neg B \rightarrow \forall x \neg A(x)$	Contrapositive
4.	$\{\exists x A(x) \rightarrow B\} \vdash \forall x(\neg B \rightarrow \neg A(x))$	Theorem 8.19
5.	$\{\exists x A(x) \rightarrow B\} \vdash \forall x(A(x) \rightarrow B)$	
6.	$\vdash \exists x A(x) \rightarrow B \rightarrow \forall x(A(x) \rightarrow B)$	Deduction

$\square$