Predicate Logic - Natural Deduction

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Natural Deduction

Proofs in the natural deduction for predicate logic are similar to those for propositional logic.

- We have new proof rules for dealing with $\forall$, $\exists$, and with the equality $(=)$ symbol
- As in the natural deduction for propositional logic, the additional rules for the quantifiers and equality will come in two flavors: introduction and elimination rules

Proof Rules Revisited

	Introduction	Elimination
$\rightarrow$	$\dfrac{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}{\phi \rightarrow \psi} \rightarrow_i$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow_e$
$\neg$	$\dfrac{\begin{array}{c}\phi \\ \vdots \\ \bot\end{array}}{\neg\phi} \neg_i$	$\dfrac{\neg\phi \quad \begin{array}{c}\neg\phi \\ \vdots \\ \psi\end{array} \quad \begin{array}{c}\neg\phi \\ \vdots \\ \neg\psi\end{array}}{\phi} \neg_e$
$\bot$	(No introduction rule for $\bot$)	$\dfrac{\bot}{\phi} \bot_e$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi} \neg\neg_e$

Proof rules for $=$

- Any term t has to be equal to itself: $\dfrac{}{t = t} =_i$.
- It becomes more powerful when we state the elimination rule: $\dfrac{t_2 = t_1 \quad \phi[t_1/x]}{\phi[t_2/x]} =_e$. This is actually a formal definition of substitution.
- This will help us with the quantifiers.

Proof Rules for $\forall$ and $\exists$

Note the boxes: they depict the scope of the dummy variables (x_0) rather than the assumption itself.

	Introduction	Elimination
$\forall$	$\dfrac{\boxed{\begin{array}{c} x_0 \; : \\ \vdots \\ \phi[x_0/x] \end{array}}}{\forall x \phi} \; \forall x_i$	$\dfrac{\forall x \phi}{\phi[t/x]} \; \forall x_e$
$\exists$	$\dfrac{\phi[t/x]}{\exists x \phi} \; \exists x_i$	$\dfrac{\exists x \phi \quad \boxed{\begin{array}{c} x_0 \; : \; \phi[x_0/x] \\ \vdots \\ \chi \end{array}}}{\chi} \; \exists x_e$

$\forall x_e, \forall x_i$

$\forall x_e$: If $\forall x \phi$ is true, then you could replace the x in ϕ by any term t.

- t must be free for x in ϕ.
- For example, consider $\phi = \exists y(x < y)$, i.e. $\forall x \exists y(x < y)$.
- Suppose that we replace x with y, which is a term that is not free in ϕ. Then, $\phi[y/x] = \exists y(y < y)$!

$\forall x_i$: If, starting with a **fresh** variable x_0, you are able to prove some formula $\phi[x_0/x]$ with x_0 in it, then (because x_0 is fresh) you can derive $\forall x \phi$.

- x_0 does **not** occur outside the box

A helpful way to approach $\forall$ is to think of it as a generalisation of $\wedge$. To prove $\forall x \phi$, one needs to show $\phi[x_i/x]$ with all x_is; to prove $\phi_1 \wedge \phi_2$, one needs to prove ϕ_i for every $i \in \{1, 2\}$.

Example

$$\forall x(P(x) \rightarrow Q(x)), \forall x P(x) \vdash \forall x Q(x)$$

1.	$\forall x(P(x) \rightarrow Q(x)$	premise
2.	$\forall x P(x)$	premise
3.	$x_0 : P(x_0) \rightarrow Q(x_0)$	$\forall x_e$, 1
4.	$P(x_0)$	$\forall x_e$, 2
5.	$Q(x_0)$	$\rightarrow_e$, 3, 4
6.	$\forall x Q(x)$	$\forall x_i$, 3-5

$\exists x_i, \exists x_e$

$\exists x_i$: It simply says that we can deduce $\exists x \phi$ whenever we have $\phi[t/x]$ for some term t

- t must be free for x in ϕ

$\exists x_e$: We know $\exists x \phi$ is true, so ϕ is true for at least one value of x

- So we do a case analysis over all those possible values, writing x_0 as a generic value representing them all. If assuming $\phi[x_0/x]$ allows us to prove χ which does not mention x_0, then χ will be true regardless of the choice of x_0.

A helpful way to approach $\exists$ is to think of it as a generalisation of $\vee$. To prove $\exists x \phi$, one needs to show $\phi[x_i/x]$ with at least one x_i; to prove $\phi_1 \vee \phi_2$, one needs to prove ϕ_i for at least one $i \in \{1, 2\}$.

Example

$\forall x(P(x) \to Q(x)), \exists x P(x) \vdash \exists x Q(x)$

$$
\begin{array}{lll}
1. & \forall x(P(x) \to Q(x) & \text{premise} \\
2. & \exists x P(x) & \text{premise} \\
3. & x_0 : P(x_0) & \text{assumption} \\
4. & P(x_0) \to Q(x_0) & \forall x_e, 1 \\
5. & Q(x_0) & \to_e, 4, 3 \\
6. & \exists x Q(x) & \exists x_i, 5 \\
7. & \exists x Q(x) & \exists x_e, 2, 3\text{-}6
\end{array}
$$

Note that $\exists x$ in the conclusion has to be introduced within the box. $\exists x Q(x)$ is the instantiation of x in rule $\exists x$ (line 2): since it does not contain an accurrence of x_0, it can leave the box (hence the repetition in the last line).

Example: strange case of escaping the scope box

$\forall x(P(x) \rightarrow Q(x), \exists x P(x) \vdash \exists x Q(x)$

1.	$\forall x(P(x) \rightarrow Q(x)$	premise
2.	$\exists x P(x)$	premise
3.	$x_0 : P(x_0)$	assumption
4.	$P(x_0) \rightarrow Q(x_0)$	$\forall x_e$, 1
5.	$Q(x_0)$	$\rightarrow_e$, 4, 3
6.	$\exists x Q(x)$	$\exists x_i$, 5
7.	$\exists x Q(x)$	$\exists x_e$, 2, 3-6

$\forall x(P(x) \rightarrow Q(x), \exists x P(x) \vdash \exists x Q(x)$

1.	$\forall x(P(x) \rightarrow Q(x)$	premise
2.	$\exists x P(x)$	premise
3.	$x_0 : P(x_0)$	assumption
4.	$P(x_0) \rightarrow Q(x_0)$	$\forall x_e$, 1
5.	$Q(x_0)$	$\rightarrow_e$, 4, 3
6.	$Q(x_0)$	$\exists x_e$, 2, 3-5
7.	$\exists x Q(x)$	$\exists x_i$, 5

Is anything wrong with this proof?

Seems to work here, but bear this in mind.

Example

$$\forall x(Q(x) \to R(x)), \exists x(P(x) \wedge Q(x)) \vdash \exists x(P(x) \wedge R(x))$$

1.	$\forall x(Q(x) \to R(x))$	premise
2.	$\exists x(P(x) \wedge Q(x))$	premise
3.	$x_0 : P(x_0) \wedge Q(x_0)$	assumption $(\exists x)$
4.	$Q(x_0) \to R(x_0)$	$\forall x_e$, 1
5.	$Q(x_0)$	$\wedge_e$ 3
6.	$R(x_0)$	$\to_e$, 4, 5
7.	$P(x_0)$	$\wedge_e$, 3
8.	$P(x_0) \wedge R(x_0)$	$\wedge_i$, 6, 7
9.	$\exists x(P(x) \wedge R(x))$	$\exists x_i$, 8
10.	$\exists x(P(x) \wedge R(x))$	$\exists x_e$, 2, 3-9

Quantifiers and Scopes

Both $\forall x_i$ and $\exists x_e$ require the dummy variable cannot occur outside the boxes in the respective rules. When rules are nested, it is better to take a different fresh name. Consider the following example.

$$\exists x P(x), \forall x \forall y (P(x) \rightarrow Q(y)) \vdash \forall y Q(y)$$

1.	$\exists x P(x)$	premise
2.	$\forall x \forall y (P(x) \rightarrow Q(y))$	premise
3.	y_0	take an arbitrary value of y
4.	$x_0 : P(x_0)$	assumption ($\exists x$)
5.	$\forall y (P(x_0) \rightarrow Q(y))$	$\forall x_e$, 2
6.	$(P(x_0) \rightarrow Q(y_0))$	$\forall y_e$, 5
7.	$Q(y_0)$	$\rightarrow_e$, 6, 4
8.	$Q(y_0)$	$\exists x_e 1, 4 - 7$
9.	$\forall y Q(y)$	$\forall y_i$, 3-8

Now consider a slightly different formula.

$\exists x P(x), \forall x(P(x) \to Q(x)) \vdash \forall y Q(y)$

1.	$\exists x P(x)$	premise
2.	$\forall x(P(x) \to Q(x))$	premise
3.	x_0	take an arbitrary value of x
4.	$x_0 : P(x_0)$	assumption ($\exists x$)
5.	$P(x_0) \to Q(x_0)$	$\forall x_e$, 2
6.	$Q(x_0)$	$\to_e$, 5, 4
7.	$Q(x_0)$	$\exists x_e$, 1, 4-6
8.	$\forall y Q(y)$	$\forall y_i$, 3-7

???

Which line is actually wrong?

$\neg\forall x\phi \vdash \exists x\neg\phi$

1.	$\neg\forall x P(x)$	premise
2.	$\neg\exists x\neg P(x)$	assumption
3.	x_0	taken an arbitrary value of x
4.	$\neg P(x_0)$	assumption
5.	$\exists x\neg P(x)$	$\exists x_i$, 4
6.	$\perp$	$\neg_e$, 5, 2
7.	$P(x_0)$	RAA, 4-6
8.	$\forall x P(x)$	$\forall x_i$, 3-7
9.	$\perp$	$\neg_e$, 8, 1
10.	$\exists x\neg P(x)$	RAA, 2-9

1.	$\exists x \neg P(x)$	premise
2.	$\forall x P(x)$	assumption
3.	x_0	
4.	$\neg P(x_0)$	assumption ($\exists x$)
5.	$P(x_0)$	$\forall x_e$, 2
6.	$\perp$	$\neg_e$, 3, 2
7.	$\perp$	$\exists x_e$, 1, 3-6
8.	$\neg \forall x P(x)$	$\neg_i$, 2-7

Exercises

Prove the following sequents.

- $\forall x(P(x) \rightarrow Q(x)) \vdash (\forall x \neg Q(x)) \rightarrow (\forall(x) \neg P(x))$
- $\forall x(P(x) \rightarrow \neg Q(x)) \vdash \neg(\exists x(P(x) \land Q(x)))$