

# Propositional Logic: Soundness and Completeness of $\mathcal{H}$

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# Goals of Logic

- To check whether a given formula  $\phi$  is valid (semantic)
- To prove a given formula  $\phi$  (syntactic)

$\models \phi$	$\equiv$	Semantic	$\equiv$	$\vdash \phi$
Semantic		Tableau		Syntactic
Methods				Methods
(Truth table)				( $\mathcal{G}$ , $\mathcal{H}$ , etc)

# Soundness and Completeness of $\mathcal{H}$

We can use what we have proved so far.

$$\begin{array}{ccccccc} \vdash \phi & \equiv & \mathcal{G} & \equiv & \text{Semantic} & \equiv & \models \phi \\ \text{Hilbert} & & & & \text{Tableau} & & \text{Semantic} \\ \text{System,} & & & & & & \text{Methods} \\ \mathcal{H} & & & & & & \text{(Truth table)} \end{array}$$

## Theorem 1 (3.34, Ben-Ari)

$\mathcal{H}$  is sound, that is, if  $\vdash A$  in  $\mathcal{H}$  then  $\models A$ .

How do we prove this? Structural Induction, that is:

- 1 Show that three axioms of  $\mathcal{H}$  are all valid, and
- 2 Show that if the premises of Modus Ponens rule is valid, then so is the conclusion.

# Soundness of $\mathcal{H}$

Show that three axioms of  $\mathcal{H}$  are all valid. To show that  $A$  is valid, we can show  $\neg A$  is not satisfiable, i.e., that the semantic tableau of  $\neg A$  is *closed*.

Axiom 1

$$\neg(A \rightarrow (B \rightarrow A))$$

$$A, \neg(B \rightarrow A)$$

$$A, B, \neg A$$

Axiom 3

$$\neg((\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B))$$

$$\neg B \rightarrow \neg A, \neg(A \rightarrow B)$$

$$\neg B \rightarrow \neg A, A, \neg B$$

$$\neg\neg B, A, B \quad \neg A, A, \neg B$$

$$B, A, \neg B$$

Tableau for Axiom 2:

$$\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))?$$

# Soundness of $\mathcal{H}$

Show that if MP is sound. We use Reductio Ad Absurdum.

Proof.

Suppose that MP is not sound. There would be a set of formulas  $\{A, A \rightarrow B, B\}$  such that  $A$  and  $A \rightarrow B$  are valid but  $B$  is not. If  $B$  is not valid, there is an interpretation  $\nu$  such that  $\nu(B) = F$ . Since  $A$  and  $A \rightarrow B$  are valid, for **any** interpretation, **including**  $\nu$ , resulting in  $\nu(A) = \nu(A \rightarrow B) = T$ . The truth table then states  $\nu(B) = T$ , which results in a contradiction in terms of our choice of  $\nu$ . □

$\mathcal{H}$  is sound.

## Theorem 2 (3.35, Ben-Ari)

$\mathcal{H}$  is complete, that is, if  $\models A$  then  $\vdash A$  in  $\mathcal{H}$ .

- Any valid formula can be proved in  $\mathcal{G}$  (Thm 3.8). We will show how a proof in  $\mathcal{G}$  can be mechanically transformed into a proof in  $\mathcal{H}$ .
- A set of formulas  $U$  is provable in  $\mathcal{G} \equiv$  a single formula  $\bigvee U$  is provable in  $\mathcal{H}$ .
- Certain axioms of  $\mathcal{G}$  are trivial:  $\{p, \neg p\}$  is an axiom in  $\mathcal{G}$ , then  $\vdash (p \vee \neg p)$  in  $\mathcal{H}$ , using Thm 3.10 ( $\vdash A \rightarrow A$ ). (Note that  $(A \rightarrow B) \leftrightarrow \neg A \vee B$ )
- How about  $\{q, \neg p, r, p, s\}$ ? Not trivial.



## Lemma 1 (3.36, Ben-Ari)

*If  $U' \subseteq U$  and  $\vdash \bigvee U'$ , then  $\vdash \bigvee U$  in  $\mathcal{H}$ .*

### Proof.

Suppose we have a proof of  $\bigvee U'$ . By repeated application of Thm 3.31, we can transform this into a proof of  $\bigvee U''$ , where  $U''$  is a permutation of the elements of  $U$  (Thm 3.31 Weakening:  $\vdash A \rightarrow A \vee B$ ). Now by repeated applications of the commutativity and associativity of disjunction, we can move the elements of  $U''$  to their proper places (Thm 3.32 is the Commutativity Rule, Thm 3.33 is the Associativity Rule). □

Now we use the induction on the structure of proofs in  $\mathcal{G}$  in order to prove the completeness of  $\mathcal{H}$ . That is, we show that for any proof in  $\mathcal{G}$ , there exists a mechanically corresponding proof in  $\mathcal{H}$ .

Proof.

**Case 1:** If  $U$  is an **axiom**, it contains a pair of complementary literals, and  $\vdash p \vee \neg p$  is provable in  $\mathcal{H}$ . By Lemma 1, this can be transformed into a proof of  $\bigvee U$ .

Lem 3.36: If  $U' \subseteq U$  and  $\vdash \bigvee U'$ , then  $\vdash \bigvee U$  in  $\mathcal{H}$ .

## Proof. Cont.

**Case 2:** If  $U$  is not an axiom in  $\mathcal{G}$ , the last step in the proof of  $G$  is the application of either  $\alpha$ - or  $\beta$ - rule.

$$\frac{\vdash U_1 \cup \{A_1, A_2\}}{\vdash U_1 \cup \{A_1 \vee A_2\}}$$

- $\alpha$ -rule:  $\vdash U_1 \cup \{A_1 \vee A_2\}$ . By the inductive hypothesis,  $\vdash (\bigvee U_1 \vee A_1) \vee A_2$  in  $\mathcal{H}$ , from which we get  $\vdash \bigvee U_1 \vee (A_1 \vee A_2)$  by associativity.

## Proof.

- $$\frac{\vdash U_1 \cup \{A_1\} \quad \vdash U_2 \cup \{A_2\}}{\vdash U_1 \cup U_2 \cup \{A_1 \wedge A_2\}}$$
 •  $\beta$ -rule:  $\vdash U_1 \cup U_2 \cup \{A_1 \wedge A_2\}$ . By the inductive hypothesis,  $\vdash \bigvee U_1 \vee A_1$  and  $\vdash \bigvee U_2 \vee A_2$  in  $\mathcal{H}$ . From these, we need to find a proof of  $\vdash \bigvee U_1 \vee \bigvee U_2 \vee (A_1 \wedge A_2)$ .

- |     |   |   |
|-----|---|---|
| 1.  | $\vdash \bigvee U_1 \vee A_1$   | Induction Hypothesis                    |
| 2.  | $\vdash \neg \bigvee U_1 \rightarrow A_1$   | $A \vee B \models \neg A \rightarrow B$ |
| 3.  | $\vdash A_1 \rightarrow (A_2 \rightarrow (A_1 \wedge A_2))$                           | Derived rule on $\wedge$                |
| 4.  | $\vdash \neg \bigvee U_1 \rightarrow (A_2 \rightarrow (A_1 \wedge A_2))$              | MP, 2, 3                                |
| 5.  | $\vdash A_2 \rightarrow (\neg \bigvee U_1 \rightarrow (A_1 \wedge A_2))$              | Exchanged Antecedents                   |
| 6.  | $\vdash \bigvee U_2 \vee A_2$   | Induction Hypothesis                    |
| 7.  | $\vdash \neg \bigvee U_2 \rightarrow A_2$   | $A \vee B \models \neg A \rightarrow B$ |
| 8.  | $\vdash \neg \bigvee U_2 \rightarrow (\neg \bigvee U_1 \rightarrow (A_1 \wedge A_2))$ | MP, 7, 5                                |
| 9.  | $\vdash \neg \bigvee U_2 \rightarrow (\bigvee U_1 \vee (A_1 \wedge A_2))$             | $A \vee B \models \neg A \rightarrow B$ |
| 10. | $\vdash \bigvee U_2 \vee (\bigvee U_1 \vee (A_1 \wedge A_2))$                         | $A \vee B \models \neg A \rightarrow B$ |
| 11. | $\vdash \bigvee U_1 \vee \bigvee U_2 \vee (A_1 \wedge A_2)$                           | Associativity                           |



# Consistency of $\mathcal{H}$

## Definition 1 (3.38, Ben-Ari)

A set of formulas  $U$  is inconsistent iff for some formula  $A$ ,  $U \vdash A$  and  $U \vdash \neg A$ .  $U$  is consistent iff  $U$  is not inconsistent.

## Theorem 3 (3.39, Ben-Ari)

*$U$  is inconsistent iff for all  $A$ ,  $U \vdash A$ .*

## Proof.

Let  $A$  be an arbitrary formula. Since  $U$  is inconsistent, for some formula  $B$ ,  $U \vdash B$  and  $U \vdash \neg B$ . By Thm 3.21,  $\vdash B \rightarrow (\neg B \rightarrow A)$ . Using MP twice,  $U \vdash A$ . The converse is trivial.  $\square$

## Corollary 1 (3.40)

*$U$  is consistent iff for some  $A$ ,  $U \not\vdash A$ .*

## Theorem 4 (3.41)

*$U \vdash A$  iff  $U \cup \{\neg A\}$  is inconsistent.*

**Variant Hilbert Systems** almost always have MP as the single rule, while having different choice of primitive operators and axioms. For example, a variant  $\mathcal{H}'$  replaces the third axiom with:

$$\text{Axiom 3': } \vdash (\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$$

## Theorem 5

*$\mathcal{H}$  and  $\mathcal{H}'$  are equivalent.*

# Variants of $\mathcal{H}$

Proof of Axiom 3' in  $\mathcal{H}$ :

1.  $\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash \neg B$  Assumption
2.  $\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash \neg B \rightarrow A$  Assumption
3.  $\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash A$  MP, 1, 2
4.  $\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash \neg B \rightarrow \neg A$  Assumption
5.  $\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash A \rightarrow B$  Contrapositive, 4
6.  $\{\neg B \rightarrow \neg A, \neg B \rightarrow A, \neg B\} \vdash B$  MP, 3, 5
7.  $\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash \neg B \rightarrow B$  Deduction, 7
8.  $\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash (\neg B \rightarrow B) \rightarrow B$  Theorem 3.29
9.  $\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash B$  MP, 8, 9
10.  $\{\neg B \rightarrow \neg A\} \vdash (\neg B \rightarrow A) \rightarrow B$  Deduction, 9
11.  $\vdash (\neg B \rightarrow \neg A) \rightarrow (\neg B \rightarrow A) \rightarrow B$  Deduction, 10



$\mathcal{H}''$  has the same MP rule, but a different set of axioms:

- 1  $\vdash A \vee A \rightarrow A$
- 2  $\vdash A \rightarrow A \vee B$
- 3  $\vdash A \vee B \rightarrow B \vee A$
- 4  $\vdash B \rightarrow C \rightarrow (A \vee B \rightarrow A \vee C)$

$\mathcal{H}'''$  has only one axiom called Meredith's Axiom:

$$(\{[(A \rightarrow B) \rightarrow (\neg C \rightarrow \neg D)] \rightarrow C\} \rightarrow E) \rightarrow [(E \rightarrow A) \rightarrow (D \rightarrow A)]$$

Arrrrgh.

# Subformula Property

## Definition 2 (3.48, Ben-Ari)

A deductive system has the **subformula property** if any formula appearing in a proof of  $A$  is either a subformula of  $A$  or the negation of a subformula of  $A$ .

$\mathcal{G}$  has the subformula property;  $\mathcal{H}$  does not. Why?

If a deductive system has the subformula property, then mechanical proof may become possible, since

- there exist only a finite number of subformulas for a finite formula  $\phi$
- there exist only a finite number of inference rules

The rest is the machine's work.

# Automated Proof

One desirable property of a deductive system is to generate an automated/mechanical proof.

- We have decision procedures to check validity of a propositional formula automatically (i.e., truth table and semantic tableau).
- Note that decision procedures require knowledge on all interpretations (i.e., infinite number of interpretations, in general) which is not feasible except for propositional logic.

A deductive proof requires only a finite set of sets of formulas, because a deductive proof system analyses the target formula only, not its interpretations.

- Many research works to develop (semi)automated theorem prover.
- No obvious connection between the formula and its proof in  $\mathcal{H}$ ; makes a proof in  $\mathcal{H}$  difficult (no mechanical proof).
- One has to rely on one's brain to select proper axioms.