

# Propositional Logic: Hilbert System, $\mathcal{H}$

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Unlike  $\mathcal{G}$ , which deals with sets of formulas,  $\mathcal{H}$  is a deductive system for single formulas. In  $\mathcal{G}$ , there is one definition of axioms, and multiple rules. In  $\mathcal{H}$ , there are many axioms, but only one rule.

## Definition 1 (3.9, Ben-Ari)

$\mathcal{H}$  is a deductive system with three axiom schemes and one rule of inference. For any formulas,  $A, B$  and  $C$ , the following formulas are axioms:

- $\vdash (A \rightarrow (B \rightarrow A))$
- $\vdash ((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$
- $\vdash (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$

$\mathcal{H}$  uses *Modus Ponens* (MP) as the single inference rule:

$$\frac{\vdash A \quad \vdash A \rightarrow B}{\vdash B} \text{ MP}$$

## Theorem 1 (3.10, Ben-Ari)

*For any formula  $\phi$ ,  $\phi \vdash \phi$ .*

### Proof.

Think  $A : \phi, B : \phi \rightarrow \phi, C : \phi$  when we refer to axiom schemes.

1.  $\vdash (\phi \rightarrow ((\phi \rightarrow \phi) \rightarrow \phi)) \rightarrow ((\phi \rightarrow (\phi \rightarrow \phi) \rightarrow (\phi \rightarrow \phi))$  Axiom 2
2.  $\vdash \phi \rightarrow ((\phi \rightarrow \phi) \rightarrow \phi)$  Axiom 1
3.  $\vdash (\phi \rightarrow (\phi \rightarrow \phi)) \rightarrow (\phi \rightarrow \phi)$  MP, 1, 2
4.  $\vdash \phi \rightarrow (\phi \rightarrow \phi)$  Axiom 1 ( $A, B : \phi$ )
5.  $\vdash \phi \rightarrow \phi$  MP, 3, 4



Note that  $\{\rightarrow, \neg\}$  is an adequate set of operators, i.e. can replace all other binary operators through semantic equivalence.

However, for the sake of expressiveness, we introduce new rules of inference, called *derived rules*, to  $\mathcal{H}$ . We then use derived rules to transform a proof into another (usually longer) proof, which uses just the original axioms and MP.

Consequently, derived rules should be proven to be *sound* with respect to  $\mathcal{H}$ . That is, the use of the derived rule does not increase the set of provable theorems in  $\mathcal{H}$ . That is, it should be possible to prove a derived rule of interest, without using itself.

# Deduction Rule

## Definition 2 (3.12)

Let  $U$  be a set of formulas, and  $A$  a formula. The notation  $U \vdash A$  means that the formulas in  $U$  are *assumptions* in the proof of  $A$ . A *proof* is a sequence of lines  $U_i \vdash \phi_i$ , such that for each  $i$ ,  $U_i \subseteq U$ , and  $\phi_i$  is an axiom, a previously proved theorem, a member of  $U_i$  or can be derived by *MP* from previous lines  $U'_i \vdash \phi'_i, U''_i \vdash \phi''_i$ , where  $i', i'' < i$ .

## Definition 3 (3.13)

**Deduction rule:** suppose you want to prove  $A \rightarrow B$ . First, we assume  $A$ , that is, treat  $A$  as if it is an additional axiom, in addition to the given ones,  $U$ . Then prove  $U \cup \{A\} \vdash B$ . This conclusion discharges our initial assumption  $A$ . That is, we have

$$\frac{U \cup \{A\} \vdash B}{U \vdash A \rightarrow B}$$

now proved that  $A \rightarrow B$ . In other words,  $U \vdash A \rightarrow B$ .

# Soundness of the Deduction Rule in $\mathcal{H}$

## Theorem 2 (3.14, Ben-Ari)

*The deduction rule is a **sound** derived rule.*

### Proof.

We show, by induction on the length  $n$  of the proof of  $U \cup A \vdash B$ , how to obtain a proof of  $U \vdash A \rightarrow B$  that does not use the deduction rule (i.e. show *soundness*).

For  $n = 1$ ,  $B$  is proved in a single step. Consequently,  $B$  is either an element of  $U \cup \{A\}$ , an axiom in  $\mathcal{H}$ , or a previously proved theorem.

- If  $B$  is actually  $A$ , then  $\vdash A \rightarrow A$  by Theorem 1, so naturally  $U \vdash A \rightarrow A$ .
- If  $B \in U$  (i.e.  $B$  is a proven theorem), or  $B$  is an axiom, then  $U \vdash B$ . Then  $B$  is proved in a single application of *MP* as follows:

- |    |  |                  |
|----|--|------------------|
| 1. | $U \vdash B$                               | Axiom or Theorem |
| 2. | $U \vdash B \rightarrow (A \rightarrow B)$ | Axiom #1         |
| 3. | $U \vdash A \rightarrow B$                 | MP, 1, 2         |

# Soundness of the Deduction Rule in $\mathcal{H}$

## Proof.

If  $n > 1$ , the last step in the proof of  $U \cup \{A\} \vdash B$  is either a one-step inference of  $B$  or an inference of  $B$  using *MP*.

In the first case, the result holds by the proof for  $n = 1$ .

Otherwise, *MP* was used, so there is a formula  $C$  and lines  $i, j < n$  in the proof such that line  $i$  in the proof is  $U \cup \{A\} \vdash C$  and line  $j$  is  $U \cup \{A\} \vdash C \rightarrow B$ . By the inductive hypothesis,  $U \vdash A \rightarrow C$  and  $U \vdash A \rightarrow (C \rightarrow B)$ . Based on these, the proof of  $U \vdash A \rightarrow B$  is given by:

- |    |  |                      |
|----|--|----------------------|
| 1. | $U \vdash A \rightarrow C$   | Inductive Hypothesis |
| 2. | $U \vdash A \rightarrow (C \rightarrow B)$   | Inductive Hypothesis |
| 3. | $U \vdash (A \rightarrow (C \rightarrow B)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow B))$ | Axiom #2             |
| 4. | $U \vdash (A \rightarrow C) \rightarrow (A \rightarrow B)$   | MP, 2, 3             |
| 5. | $U \vdash A \rightarrow B$   | MP, 1, 4             |



# Derived Rules in $\mathcal{H}$

## Theorem 3 (3.16, Ben-Ari)

$$\vdash (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$$

## Proof.

- |    |  |              |
|----|--|--------------|
| 1. | $\{A \rightarrow B, B \rightarrow C, A\} \vdash A$                                       | Assumption   |
| 2. | $\{A \rightarrow B, B \rightarrow C, A\} \vdash A \rightarrow B$                         | Assumption   |
| 3. | $\{A \rightarrow B, B \rightarrow C, A\} \vdash B$                                       | MP, 1, 2     |
| 4. | $\{A \rightarrow B, B \rightarrow C, A\} \vdash B \rightarrow C$                         | Assumption   |
| 5. | $\{A \rightarrow B, B \rightarrow C, A\} \vdash C$                                       | MP, 3, 4     |
| 6. | $\{A \rightarrow B, B \rightarrow C\} \vdash A \rightarrow C$                            | Deduction, 5 |
| 7. | $\{A \rightarrow B\} \vdash ((B \rightarrow C) \rightarrow (A \rightarrow C))$           | Deduction, 6 |
| 8. | $\vdash (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ | Deduction, 7 |

□

## Definition 4 (Rule of Transitivity)

$$\frac{U \vdash A \rightarrow B \quad U \vdash B \rightarrow C}{U \vdash A \rightarrow C}$$



# Derived Rules in $\mathcal{H}$

## Theorem 4

$$\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$$

## Proof.

- |    |  |              |
|----|--|--------------|
| 1. | $\{A \rightarrow (B \rightarrow C), B, A\} \vdash A$                                     | Assumption   |
| 2. | $\{A \rightarrow (B \rightarrow C), B, A\} \vdash A \rightarrow (B \rightarrow C)$       | Assumption   |
| 3. | $\{A \rightarrow (B \rightarrow C), B, A\} \vdash B \rightarrow C$                       | MP, 1, 2     |
| 4. | $\{A \rightarrow (B \rightarrow C), B, A\} \vdash B$                                     | Assumption   |
| 5. | $\{A \rightarrow (B \rightarrow C), B, A\} \vdash C$                                     | MP, 4, 3     |
| 6. | $\{A \rightarrow (B \rightarrow C), B\} \vdash A \rightarrow C$                          | Deduction, 5 |
| 7. | $\{A \rightarrow (B \rightarrow C)\} \vdash B \rightarrow (A \rightarrow C)$             | Deduction, 6 |
| 8. | $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$ | Deduction, 7 |

□

## Definition 5 (Rule of Exchanged Antecedent)

$$\frac{U \vdash A \rightarrow (B \rightarrow C)}{U \vdash B \rightarrow A \rightarrow C}$$

# Theorems for other operators in $\mathcal{H}$

## Theorem 5

$$\vdash A \rightarrow (B \rightarrow (A \wedge B))$$

## Proof.

1.  $\{A, B\} \vdash (A \rightarrow \neg B) \rightarrow (A \rightarrow \neg B)$  Theorem 1
2.  $\{A, B\} \vdash A \rightarrow ((A \rightarrow \neg B) \rightarrow \neg B)$  Exchange of Antecedent
3.  $\{A, B\} \vdash A$  Assumption
4.  $\{A, B\} \vdash (A \rightarrow \neg B) \rightarrow \neg B$  MP, 3, 2
5.  $\{A, B\} \vdash \neg \neg B \rightarrow \neg(A \rightarrow \neg B)$  Contrapositive
6.  $\{A, B\} \vdash B$  Assumption
7.  $\{A, B\} \vdash \neg \neg B$  Double Negation
8.  $\{A, B\} \vdash \neg(A \rightarrow \neg B)$  MP, 5, 7
9.  $\{A\} \vdash B \rightarrow \neg(A \rightarrow \neg B)$  Deduction
10.  $\vdash A \rightarrow (B \rightarrow \neg(A \rightarrow \neg B))$  Deduction
11.  $\vdash A \rightarrow (B \rightarrow (A \wedge B))$   $A \wedge B \models \neg(A \rightarrow \neg B)$



Prove the following theorems in  $\mathcal{H}$ :

- $\vdash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$  (Contraposition)
- $\vdash \neg\neg A \rightarrow A$  (Double Negation)
- given that  $\vdash \text{true}$  and  $\vdash \neg\text{false}$ , prove  $\vdash (\neg A \rightarrow \text{false}) \rightarrow A$  (Reductio Ad Absurdum)