Propositional Logic: Hilbert System, $\mathcal{H}$

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Hilbert System, $\mathcal{H}$

Unlike $\mathcal{G}$, which deals with sets of formulas, $\mathcal{H}$ is a deductive system for single formulas. In $\mathcal{G}$, there is one definition of axioms, and multiple rules. In $\mathcal{H}$, there are many axioms, but only one rule.

Definition 1 (3.9, Ben-Ari)

$\mathcal{H}$ is a deductive system with three axiom schemes and one rule of inference. For any formulas, A, B and C, the following formulas are axioms:

- $\vdash (A \rightarrow (B \rightarrow A))$
- $\vdash ((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
- $\vdash (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$

$\mathcal{H}$ uses *Modus Ponens* (MP) as the single inference rule:

$$\frac{\vdash A \quad \vdash A \rightarrow B}{\vdash B}\ MP$$

Hilbert System, $\mathcal{H}$

Theorem 1 (3.10, Ben-Ari)

For any formula ϕ, $\phi \vdash \phi$.

Proof.

Think $A : \phi, B : \phi \to \phi, C : \phi$ when we refer to axiom schemes.

1.	$\vdash (\phi \to ((\phi \to \phi) \to \phi)) \to ((\phi \to (\phi \to \phi) \to (\phi \to \phi))$	Axiom 2
2.	$\vdash \phi \to ((\phi \to \phi) \to \phi)$	Axiom 1
3.	$\vdash (\phi \to (\phi \to \phi)) \to (\phi \to \phi)$	MP, 1, 2
4.	$\vdash \phi \to (\phi \to \phi)$	Axiom 1 ($A, B : \phi$)
5.	$\vdash \phi \to \phi$	MP, 3, 4

$\square$

Hilbert System, $\mathcal{H}$

Note that $\{\rightarrow, \neg\}$ is an adequate set of opeartors, i.e. can replace all other binary operators through semantic equivalence.

However, for the sake of expressiveness, we introduce new rules of inference, called *derived rules*, to $\mathcal{H}$. We then use use derived rules to transform a proof into another (usually longer) proof, which uses just the original axioms and MP.

Consequently, derived rules should be proven to be *sound* with respect to $\mathcal{H}$. That is, the use of the derived rule does not increase the set of provable theorems in $\mathcal{H}$. That is, it should be possible to prove a derived rule of interest, without using itself.

Deduction Rule

Definition 2 (3.12)

Let U be a set of formulas, and A a formula. The notation $U \vdash A$ means that the formulas in U are *assumptions* in the proof of A. A *proof* is a sequence of lines $U_i \vdash \phi_i$, such that for each i, $U_i \subseteq U$, and ϕ_i is an axiom, a previously proved theorem, a member of U_i or can be derived by MP from previous lines $U_i' \vdash \phi_i'$, $U_i'' \vdash \phi_i''$, where $i', i'' < i$.

Definition 3 (3.13)

Deduction rule: suppose you want to prove $A \rightarrow B$. First, we assume A, that is, treat A as if it is an additional axiom, in addition to the given ones, U. Then prove $U \cup \{A\} \vdash B$. This conclusion discharges our initial assumption A. That is, we have now proved that $A \rightarrow B$. In other words, $\dfrac{U \cup \{A\} \vdash B}{U \vdash A \rightarrow B}$.

Soundness of the Deduction Rule in $\mathcal{H}$

Theorem 2 (3.14, Ben-Ari)

The deduction rule is a **sound** *derived rule.*

Proof.

We show, by induction on the length n of the proof of $U \cup A \vdash B$, how to obtain a proof of $U \vdash A \rightarrow B$ that does not use the deduction rule (i.e. show *soundness*).

For $n = 1$, B is proved in a single step. Consequently, B is either an element of $U \cup \{A\}$, an axiom in $\mathcal{H}$, or a previously proved theorem.

- If B is actually A, then $\vdash A \rightarrow A$ by Theorem 1, so naturally $U \vdash A \rightarrow A$.
- If $B \in U$ (i.e. B is a proven theorem), or B is an axiom, then $U \vdash B$. Then B is proved in a single application of MP as follows:

1.	$U \vdash B$	Axiom or Theorem
2.	$U \vdash B \rightarrow (A \rightarrow B)$	Axiom #1
3.	$U \vdash A \rightarrow B$	MP, 1, 2

Soundness of the Deduction Rule in $\mathcal{H}$

Proof.

If $n > 1$, the last step in the proof of $U \cup \{A\} \vdash B$ is either a one-step inference of B or an inference of B using MP.

In the first case, the result holds by the proof for n $= 1$.

Otherwise, MP was used, so there is a formula C and lines $i, j < n$ in the proof such that line i in the proof is $U \cup \{A\} \vdash C$ and line j is $U \cup \{A\} \vdash C \rightarrow B$. By the inductive hypothesis, $U \vdash A \rightarrow C$ and $U \vdash A \rightarrow (C \rightarrow B)$. Based on these, the proof of $U \vdash A \rightarrow B$ is given by:

1.	$U \vdash A \rightarrow C$	Inductive Hypothesis
2.	$U \vdash A \rightarrow (C \rightarrow B)$	Inductive Hypothesis
3.	$U \vdash (A \rightarrow (C \rightarrow B)) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow B))$	Axiom #2
4.	$U \vdash (A \rightarrow C) \rightarrow (A \rightarrow B)$	MP, 2, 3
5.	$U \vdash A \rightarrow B$	MP, 1, 4

□

Theorem 3 (3.16, Ben-Ari)

$$\vdash (A \to B) \to ((B \to C) \to (A \to C))$$

Proof.

1.	$\{A \to B, B \to C, A\} \vdash A$	Assumption
2.	$\{A \to B, B \to C, A\} \vdash A \to B$	Assumption
3.	$\{A \to B, B \to C, A\} \vdash B$	MP, 1, 2
4.	$\{A \to B, B \to C, A\} \vdash B \to C$	Assumption
5.	$\{A \to B, B \to C, A\} \vdash C$	MP, 3, 4
6.	$\{A \to B, B \to C\} \vdash A \to C$	Deduction, 5
7.	$\{A \to B\} \vdash ((B \to C) \to (A \to C))$	Deduction, 6
8.	$\vdash (A \to B) \to ((B \to C) \to (A \to C))$	Deduction, 7

$\square$

Definition 4 (Rule of Transtivity)

$$\frac{U \vdash A \to B \quad U \vdash B \to C}{U \vdash A \to C}$$

Derived Rules in $\mathcal{H}$

Theorem 4

$$\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$$

Proof.

1.	$\{A \rightarrow (B \rightarrow C), B, A\} \vdash A$	Assumption
2.	$\{A \rightarrow (B \rightarrow C), B, A\} \vdash A \rightarrow (B \rightarrow C)$	Assumption
3.	$\{A \rightarrow (B \rightarrow C), B, A\} \vdash B \rightarrow C$	MP, 1, 2
4.	$\{A \rightarrow (B \rightarrow C), B, A\} \vdash B$	Assumption
5.	$\{A \rightarrow (B \rightarrow C), B, A\} \vdash C$	MP, 4, 3
6.	$\{A \rightarrow (B \rightarrow C), B\} \vdash A \rightarrow C$	Deduction, 5
7.	$\{A \rightarrow (B \rightarrow C)\} \vdash B \rightarrow (A \rightarrow C)$	Deduction, 6
8.	$\vdash (A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$	Deduction, 7

$\square$

Definition 5 (Rule of Exchanged Antecedent)

$$\frac{U \vdash A \rightarrow (B \rightarrow C)}{U \vdash B \rightarrow A \rightarrow C}$$

Theorems for other operators in $\mathcal{H}$

Theorem 5

$$\vdash A \rightarrow (B \rightarrow (A \wedge B))$$

Proof.

1.	$\{A, B\} \vdash (A \rightarrow \neg B) \rightarrow (A \rightarrow \neg B)$	Theorem 1
2.	$\{A, B\} \vdash A \rightarrow ((A \rightarrow \neg B) \rightarrow \neg B)$	Exchange of Antecedent
3.	$\{A, B\} \vdash A$	Assumption
4.	$\{A, B\} \vdash (A \rightarrow \neg B) \rightarrow \neg B$	MP, 3, 2
5.	$\{A, B\} \vdash \neg\neg B \rightarrow \neg(A \rightarrow \neg B)$	Contrapositive
6.	$\{A, B\} \vdash B$	Assumption
7.	$\{A, B\} \vdash \neg\neg B$	Double Negation
8.	$\{A, B\} \vdash \neg(A \rightarrow \neg B)$	MP, 5, 7
9.	$\{A\} \vdash B \rightarrow \neg(A \rightarrow \neg B)$	Deduction
10.	$\vdash A \rightarrow (B \rightarrow \neg(A \rightarrow \neg B))$	Deduction
11.	$\vdash A \rightarrow (B \rightarrow (A \wedge B))$	$A \wedge B \models \neg(A \rightarrow \neg B)$

$\square$

Exercise

Prove the following theorems in $\mathcal{H}$:

- $\vdash (A \to B) \to (\neg B \to \neg A)$ (Contraposition)
- $\vdash \neg\neg A \to A$ (Double Negation)
- given that $\vdash \mathit{true}$ and $\vdash \neg \mathit{false}$, prove $\vdash (\neg A \to \mathit{false}) \to A$ (Reductio Ad Absurdum)