Propositional Logic: Semantics (2/3)

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Overview

- Logical Equivalence and Substitution
- Satisfiability, Validity, and Consequence

Logical Equivalence

Definition 1 (2.26)

Let $A_1, A_2 \in \mathscr{F}$. If $\nu_{\mathscr{I}}(A_1) = \nu_{\mathscr{I}}(A_2)$ for *all* interpretations $\mathscr{I}$, then A_1 is *logically equivalent* to A_2, denoted $A_1 \equiv A_2$.

$\mathscr{I}(p)$	$\mathscr{I}(q)$	$\nu_{\mathscr{I}}(p \vee q)$	$\nu_{\mathscr{I}}(q \vee p)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

Logical Equivalence

We can extend the result of the previous example from atomic propositions to general formulas.

Theorem 1 (2.28)

Let A_1 and A_2 be any formulas. Then $A_1 \vee A_2 \equiv A_2 \vee A_1$.

Proof.

1. Let $\mathscr{I}$ be an arbitrary interpretation for $A_1 \vee A_2$. Then, $\mathscr{I}$ is also an interpretation for $A_2 \vee A_1$, because $\mathscr{P}_{A_1} \cup \mathscr{P}_{A_2} = \mathscr{P}_{A_2} \cup \mathscr{P}_{A_1}$.
2. Similarly, $\mathscr{I}$ is an interpretation for A_1 and A_2.
3. Therefore, $\nu_{\mathscr{I}}(A_1 \vee A_2) = T \leftrightarrow (\nu_{\mathscr{I}}(A_1) = T \vee \nu_{\mathscr{I}}(A_2) = T) \leftrightarrow \nu(A_2 \vee A_1) = T$.

$\square$

Logical Equivalence

Theorem 2 (2.29)

$A_1 \equiv A_2$ if and only if $A_1 \leftrightarrow A_2$ is true in every interpretation.

- **Object Language:** the language we set out to study, i.e. propositional logic in our current case.
- **Metalanguage:** the language that is used to discuss an object language.

What is the difference between $\leftrightarrow$ and $\equiv$?

- **Material Equivalence** ($\leftrightarrow$): just another statement in the object language; truth value depends on the specific interpretation.
- **Logical Equivalance** ($\equiv$): semantic statement, i.e. if p is logically equivalent to q, it means that under every possible interpretation, p and q logically means the same thing. This is a statement in the metalanguage.

Logical Substitution

Logical equivalance justifies *substitution* of one formula for another that is equivalent.

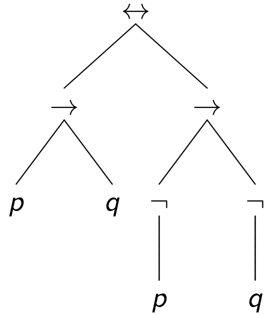
Let us present the intermediate steps first.

Definition 2 (2.30)

A is subformula of B if the formation tree for A occurs as a subtree of the formation tree for B. A is *proper* subformula of B if A is a subformula of B, but A is not identical to B.

Example 1 (2.31)

The formula $(p \to q) \leftrightarrow (\neg p \to \neg q)$ contains the following proper subformulas: $p \to q, \neg p \to \neg q, \neg p, \neg q, p$ and q



Definition 3 (2.32)

If A is a subformula of B, and A' is an arbitrary formula, then B', the *substitution* of A' for A in B, denoted $B\{A \leftarrow A'\}$, is the formula obtained by replacing all occurrences of the subtree for A in B by the tree for A'.

Theorem 3 (2.34)

Let A be a subformula of B and let A' be a formula such that $A \equiv A'$. Then $B \equiv B\{A \leftarrow A'\}$.

Substitution can be naturally used to *simplify* formulas.

$$p \wedge (\neg p \vee q) \equiv (p \wedge \neg p) \vee (p \wedge q) \equiv \mathit{false} \vee (p \wedge q) \equiv p \wedge q$$

Logical Equivalence

Definition 4 (2.35)

A binary operator, o, is *defined from* a set of operators, $O = \{o_1, \ldots, o_n\}$ iff there is a logical equivalence $A_1 \; o \; A_2 \equiv A$ where A is a formula constructed from occurrences of A_1, A_2, and operators in O.

Similarly, an unary operator o is *defined from* a set of operators, $O = \{o_1, \ldots, o_n\}$ iff there is a logical equivalence $o \; A_1 \equiv A$ where A is a formula constructed from occurrences of A_1, and operator o.

Example 2

- $\leftrightarrow$ is defined from $\{\rightarrow, \wedge\}$ because $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$.
- $\rightarrow$ is defined from $\{\neg, \vee\}$ because $A \rightarrow B \equiv \neg A \vee B$.
- $\wedge$ is defined from $\{\neg, \vee\}$ because $A \wedge B \equiv \neg(\neg A \vee \neg B)$.

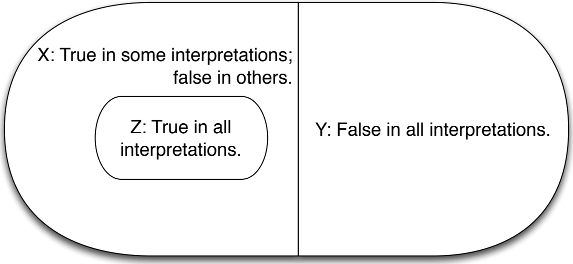
Satisfiability, Validity, and Consequences

Definition 5 (2.38)

- A propositional formula A is *satisfiable* iff $\nu_{\mathscr{I}}(A) = T$ for *some* interpretation $\mathscr{I}$.
- A satisfying interpretation is called a *model* for A.
- A is *valid*, denoted $\models A$, iff $\nu_{\mathscr{I}}(A) = T$ for *all* interpretation $\mathscr{I}$.
- A valid propositional formula is also called a *tautology*.
- A is *unsatisfiable* if and only if it is not satisfiable, that is, if $\nu_{\mathscr{I}}(A) = F$ for *all* interpretations $\mathscr{I}$.
- A is falsifiable, denoted $\not\models A$, if and only if it is not valid, that is, if $\nu_{\mathscr{I}}(A) = F$ for *some* interpretaion $\mathscr{I}$.

Theorem 4 (2.39)

A is valid iff $\neg A$ is unsatisfiable. A is satisfiable iff $\neg A$ is falsifiable.



- X (and, therefore, Z): Satisfiable.
- Y: Unsatisfiable.
- Z: Valid.
- $(X - Z) \bigcup Y$: Falsifiable (i.e. can be shown to be false).

Definition 6 (2.40)

Let $\mathscr{U} \subseteq \mathscr{F}$ be a set of formulas. An algorithm is a *decision procedure* for $\mathscr{U}$ if given an arbitrary formula $A \in \mathcal{F}$, it terminates and return the answer 'yes' if $A \in \mathscr{U}$ and the answer 'no' if $A \notin \mathscr{U}$.

By Theorem 2.39, a decision procedure for satisfiability can be used as a decision procedure for validity. Let $\mathcal{V}$ be the set of all satisfiable formulas. To decide the validity of A, we can apply the decision procesure for satisfiability of $\neg A$. This decision procedure is called a *refutation procedure*.

Example 3

Is $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow \neg q)$ valid?

Example 4

$p \vee q$ is satisfiable but not valid.

Logical Consequence

Definition 7 (2.42)

Extension of satisfiability from a single formula to a set of formulas: a set of formulas $U = A_1, \ldots, A_n$ is (*simultaneously*) *satisfiable* iff there exists an interpretation $\mathscr{I}$ such that $\nu_{\mathscr{I}}(A_1) = \ldots = \nu_{\mathscr{I}}(A_n) = T$. The satisfying interpretation is called a *model* of U. U is *unsatisfiable* iff for every interpretation $\mathscr{I}$, there exists an i such that $\nu_{\mathscr{I}}(A_i) = F$.

Logical Consequence

Definition 8 (2.48)

Let U be a set of formulas and A a formula. A is a *logical consequence* of U, denoted $U \models A$, iff every model of U is a model of A.

Theorem 5 (2.50)

$U \models A$ iff $A_1 \wedge A_2 \ldots \wedge A_n \rightarrow A$, where $U = \{A_1, \ldots, A_n\}$.

If $U = \emptyset$, the logical consequence is the same as the validity.

Theories

Logical consequence is the central concept in the foundations of mathematics; validity is often trivial and not very interesting. For example, Euclidean geometry is an extensive set of logical consequences, all deduced from the five axioms.

Definition 9 (2.55)

Let $\mathscr{T}$ be a set of formulas. $\mathscr{T}$ is *closed under logical consequence* iff for all formulas A, if $\mathscr{T} \models A$ then $A \in \mathscr{T}$. A set of formulas that is closed under logical consequence is a *theory*. The elements of $\mathscr{T}$ are theorems.

Definition 10

Let $\mathscr{T}$ be a theory. $\mathscr{T}$ is said to be *axiomatizable* iff there exists a set of formulas U such that $\mathscr{T} = \{A | U \models A\}$. The set of formulas U are the axioms of $\mathscr{T}$. If U is finite, $\mathscr{T}$ is said to be *finitely axiomatizable*.

Examples of Theory

	p	q	r	$p \vee q \vee r$	$q \rightarrow r$	$r \rightarrow p$
$\mathscr{I}_1$	T	T	T	T	T	T
$\mathscr{I}_2$	T	T	F	T	F	T
$\mathscr{I}_3$	T	F	T	T	T	T
$\mathscr{I}_4$	T	F	F	T	T	T
$\mathscr{I}_5$	F	T	T	T	T	F
$\mathscr{I}_6$	F	T	F	T	F	T
$\mathscr{I}_7$	F	F	T	T	T	F
$\mathscr{I}_8$	T	F	F	F	T	T

- $U = \{p \vee q \vee r, q \rightarrow r, r \rightarrow p\}$
- Interpretation ν_1, ν_3, ν_4 are models of U (i.e. interpretations that make all formulas in U true).
- Which of the following are true?
  1. $U \models p$
  2. $U \models q \rightarrow r$
  3. $U \models r \vee \neg q$
  4. $U \models p \wedge \neg q$

Examples of Theory

	p	q	r	$p \vee q \vee r$	$q \rightarrow r$	$r \rightarrow p$
ν_1	T	T	T	T	T	T
ν_2	T	T	F	T	F	T
ν_3	T	F	T	T	T	T
ν_4	T	F	F	T	T	T
ν_5	F	T	T	T	T	F
ν_6	F	T	F	T	F	T
ν_7	F	F	T	T	T	F
ν_8	T	F	F	F	T	T

Theory of $U = \{p \vee q \vee r, q \rightarrow r, r \rightarrow p\}$, i.e. $\mathcal{T}$ (U):

- $U \subseteq \mathcal{T}(U)$ because for all formula $A \in \mathcal{F}$, $A \models A$.
- $p \in \mathcal{T}(U)$ because $U \models p$.
- $(q \rightarrow r) \in \mathcal{T}(U)$ because $U \models (q \rightarrow r)$.
- $p \wedge (q \rightarrow r) \in \mathcal{T}(U)$ because $U \models p \wedge (q \rightarrow r)$.

Theory of Euclidean Geometry is based on the set of 5 axioms, $U = A_1, A_2, A_3, A_4, A_5$ such that:

- A_1: Any two points can be joined by a unique straight line.
- A_2: Any straight line segment can be extended indefinitely in a straight line.
- A_3: Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
- A_4: All right angles are congruent.
- A_5: For every line l and for every point P that does not lie on l, there exists a unique line m through P that is parallel to l.

The ancient Greeks suspected whether A_5 is a logical consequence of the other four. For about 2,000 years, various mathematicians tried to show $\{A_1, \ldots, A_4\} \models A_5$. Only in 1868, Beltrami showed that A_5 is independent from the rest. In other words, we accept A_5 as an axiom.

Beltrami also showed that non-Euclidean geometry (i.e. U with A_5 replaced with alternatives) is *consistent*.