Quiz on Thursday, 6th April: 15 minutes, two questions.
In Natural Deduction, each line in the proof consists of exactly one proposition. That is, $A_1, A_2, \ldots, A_n \vdash B$.

In Sequent calculus, each line in the proof consists of zero or more propositions. That is, $A_1, A_2, \ldots, A_n \vdash B_1, B_2, \ldots, B_k$. The standard semantic is, “whenever every $A_i$ is true, at least one $B_j$ will also be true”.
Axioms in $\mathcal{G}$

**Definition 1 (3.2, Ben-Ari)**

An axiom of $\mathcal{G}$ is a set of literals $U$ containing a complementary pair.

Note that sets in $\mathcal{G}$ are implicitly disjunctive. For example, $\{\neg p, q, p\}$ is an axiom, i.e. $\vdash \neg p, q, p$ in $\mathcal{G}$.
Definition 2 (3.2, Ben-Ari)

There are two types of inference rules, defined with reference to tables below:

- Let $\{\alpha_1, \alpha_2\} \subseteq U_1$ and let $U_1' = U_1 - \{\alpha_1, \alpha_2\}$. Then $U = U_1' \cup \{\alpha\}$ can be inferred.

- Let $\{\beta_1\} \subseteq U_1$, $\{\beta_2\} \subseteq U_2$ and let $U_1' = U_1 - \{\beta_1\}$, $U_2' = U_2 - \{\beta_2\}$. Then $U = U_1' \cup U_2' \cup \{\beta\}$ can be inferred.
Inference Rules in $G$

\[
\frac{\vdash U'_1 \cup \{\alpha_1, \alpha_2\}}{\vdash U'_1 \cup \{\alpha\}} \alpha
\]

\[
\vdash U'_1 \cup \{\alpha\} \vdash U'_1 \cup \{\alpha\}
\]

\[
\alpha \quad | \quad \alpha_1 \quad | \quad \alpha_2
\]

\[
\neg \neg \alpha \quad | \quad \alpha
\]

\[
\neg (\alpha_1 \land \alpha_2) \quad | \quad \neg \alpha_1 \quad | \quad \neg \alpha_2
\]

\[
\alpha_1 \lor \alpha_2 \quad | \quad \alpha_1 \quad | \quad \alpha_2
\]

\[
\alpha_1 \rightarrow \alpha_2 \quad | \quad \neg \alpha_1 \quad | \quad \alpha_2
\]

\[
\alpha_1 \uparrow \alpha_2 \quad | \quad \neg \alpha_1 \quad | \quad \neg \alpha_2
\]

\[
\neg (\alpha_1 \downarrow \alpha_2) \quad | \quad \alpha_1 \quad | \quad \alpha_2
\]

\[
\neg (\alpha_1 \leftrightarrow \alpha_2) \quad | \quad \neg (\alpha_1 \rightarrow \alpha_2) \quad | \quad \neg (\alpha_2 \rightarrow \alpha_1)
\]

\[
\alpha_1 \oplus \alpha_2 \quad | \quad \neg (\alpha_1 \rightarrow \alpha_2) \quad | \quad \neg (\alpha_2 \rightarrow \alpha_1)
\]

That is, $\alpha-$rules build up disjunctions.

\[
\frac{\vdash U'_1 \cup \{\beta_1\}}{\vdash U'_1 \cup \{\beta\}} \beta
\]

\[
\vdash U'_1 \cup \{\beta\} \vdash U'_2 \cup \{\beta\}
\]

\[
\beta \quad | \quad \beta_1 \quad | \quad \beta_2
\]

\[
\beta_1 \land \beta_2 \quad | \quad \beta_1 \quad | \quad \beta_2
\]

\[
\neg (\beta_1 \lor \beta_2) \quad | \quad \neg \beta_1 \quad | \quad \neg \beta_2
\]

\[
\neg (\beta_1 \rightarrow \beta_2) \quad | \quad \beta_1 \quad | \quad \neg \beta_2
\]

\[
\neg (\beta_1 \uparrow \beta_2) \quad | \quad \beta_1 \quad | \quad \beta_2
\]

\[
\beta_1 \downarrow \beta_2 \quad | \quad \neg \beta_1 \quad | \quad \neg \beta_2
\]

\[
\beta_1 \leftrightarrow \beta_2 \quad | \quad \beta_1 \rightarrow \beta_2 \quad \beta_2 \rightarrow \beta_1
\]

\[
\neg (\beta_1 \oplus \beta_2) \quad | \quad \beta_1 \rightarrow \beta_2 \quad \beta_2 \rightarrow \beta_1
\]

That is, $\beta-$rules build up conjunctions (consider $(a \lor b) \land (c \lor d) \models a \lor c \lor (b \land d)$).
Example Proof

Prove that $\vdash p \lor (q \land r) \rightarrow (p \lor q) \land (p \lor r)$ in $G$.

1. $\vdash \neg p, p, q$  \hspace{1cm} Axiom
2. $\vdash \neg p, (p \lor q)$  \hspace{1cm} $\alpha \lor$, 1
3. $\vdash \neg p, p, r$  \hspace{1cm} Axiom
4. $\vdash \neg p, (p \lor r)$  \hspace{1cm} $\alpha \lor$, 3
5. $\vdash \neg p, (p \lor q) \land (p \lor r)$  \hspace{1cm} $\beta \land$, 2, 4
6. $\vdash \neg q, \neg r, p, q$  \hspace{1cm} Axiom
7. $\vdash \neg q, \neg r, (p \lor q)$  \hspace{1cm} $\alpha \lor$, 6
8. $\vdash \neg q, \neg r, p, r$  \hspace{1cm} Axiom
9. $\vdash \neg q, \neg r, (p \lor r)$  \hspace{1cm} $\alpha \lor$, 8
10. $\vdash \neg q, \neg r, (p \lor q) \land (p \lor r)$  \hspace{1cm} $\beta \land$, 7, 9
11. $\vdash \neg (q \land r), (p \lor q) \land (p \lor r)$  \hspace{1cm} $\alpha \land$, 10
12. $\vdash \neg (p \lor (q \land r)), (p \lor q) \land (p \lor r)$  \hspace{1cm} $\beta \lor$, 5, 11
13. $\vdash p \lor (q \land r) \rightarrow (p \lor q) \land (p \lor r)$  \hspace{1cm} $\alpha \rightarrow$, 12
How do you magically come up with the axioms \( \{\neg p, p, q\} \), \( \{\neg p, p, r\} \), \( \{\neg q, \neg r, p, q\} \), and \( \{\neg q, \neg r, p, r\} \)?

Haven’t we seen something like this before?
\[
\vdash (p \lor q) \rightarrow (q \lor p)
\]

Proof in \( \mathcal{G} \)

\[
\neg p, q, p \quad \neg q, q, p
\]

\[
\neg (p \lor q), q, p
\]

\[
\neg (p \lor q), (q \lor p)
\]

\[
(p \lor q) \rightarrow (q \lor p)
\]

\[
\neg ((p \lor q) \rightarrow (q \lor p))
\]

\[
(p \lor q), \neg (q \lor p)
\]

\[
(p \lor q), \neg q, \neg p
\]

\[
p, \neg q, \neg p \quad q, \neg q, \neg p
\]

\[
\text{UNSAT} \quad \text{UNSAT}
\]

Semantic Tableau

(Set are conjunctive)
Theorem 1 (3.6, Ben-Ari)

Let $A$ be a formula in propositional logic. Then $\vdash A$ in $\mathcal{G}$ if and only if there is a closed semantic tableau for $\neg A$.

Theorem 2 (3.7, Ben-Ari)

Let $U$ be a set of formulas and let $\bar{U}$ be the set of complements of formulas in $U$. Then, $\vdash U$ in $\mathcal{G}$ if and only if there is a closed semantic tableau for $\bar{U}$.
We prove that, if there exists a closed semantic tableau for $\bar{U}$, then $\vdash U$ in $\mathcal{G}$. The opposite direction is left for you.

Proof.

Let $\mathcal{T}$ be a closed semantic tableau for $\bar{U}$. We prove $\vdash U$ by induction on $h$, the height of $\mathcal{T}$.

- If $h = 0$, then $\mathcal{T}$ consists of a single node labeled by $\bar{U}$. By assumption, $\mathcal{T}$ is closed, so it contains a complementary pair of literals $\{p, \neg p\}$, that is, $\bar{U} = \bar{U}' \cup \{p, \neg p\}$. Obviously, $U = U' \cup \{-p, p\}$ is an axiom in $\mathcal{G}$, hence $\vdash U$. 
Proof. Cont.

- If $h > 0$, then some tableau rule was used on an $\alpha$- or $\beta$-formula at the root of $T$ on a formula $\phi \in \bar{U}$, that is, $\bar{U} = \bar{U}' \cup \bar{\phi}$. The proof proceeds by cases, where you must be careful to distinguish between applications of the tableau rules and applications of the Gentzen rules of the same name.

- Case 1: $\phi$ is an $\alpha$-formula (such as) $\neg(A_1 \lor A_2)$. The tableau rule created a child node labeled by the set of formulas $\bar{U}' \cup \{\neg A_1, \neg A_2\}$. By assumption, the subtree rooted at this node is a closed tableau, so by the inductive hypothesis, $\models U' \cup \{A_1, A_2\}$. Using the appropriate rule of inference from $\mathcal{G}$, we obtain $\models U' \cup \{A_1 \lor A_2\}$, that is, $\models U' \cup \{\phi\}$, which is $\models U$.  

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Propositional Logic: Gentzen System, $\mathcal{G}$
Proof.

- If $h > 0$, then some tableau rule was used on an $\alpha$- or $\beta$-formula at the root of $T$ on a formula $\phi \in \bar{U}$, that is, $\bar{U} = \bar{U}' \cup \bar{\phi}$. The proof proceeds by cases, where you must be careful to distinguish between applications of the tableau rules and applications of the Gentzen rules of the same name.

- Case 2: $\phi$ is a $\beta$-formula (such as) $\neg(B_1 \land B_2)$. The tableau rule created two child nodes labeled by the sets of formulas $\bar{U}' \cup \{\neg B_1\}$ and $\bar{U}' \cup \{\neg B_2\}$. By assumption, the subtrees rooted at this node are closed, so by the inductive hypothesis $\vdash U' \cup \{B_1\}$ and $\vdash U' \cup \{B_2\}$. Using the appropriate rule of inference from $\mathcal{G}$, we obtain $\vdash U' \cup \{B_1 \land B_2\}$, that is, $\vdash U' \cup \{\phi\}$, which is $\vdash U$.  

\[ \square \]
Why $G$ and not natural deduction?

Taste. Or, more appropriately, aesthetics.

Natural deduction feels more, umm, natural. It is also more simplistic; having multiple disjunct on the right hand side, in $G$, is clearly cumbersome and adds complexity.

$G$ shows the symmetric nature of negation more vividly.

\[
A_1, \ldots, A_n \vdash B_1, \ldots, B_k
\]
\[\vdash (A_1 \land \cdots \land A_n) \rightarrow (B_1 \lor \cdots \lor B_k)\]
\[\vdash \neg A_1 \lor \neg A_2 \lor \cdots \lor \neg A_n \lor B_1 \lor B_2 \lor \cdots \lor B_k\]
\[\vdash \neg (A_1 \land A_2 \land \cdots \land A_n \land \neg B_1 \land \neg B_2 \land \cdots \land \neg B_k)\]
Theorem 3 (3.8 in Ben-Ari)
\[ \models A \text{ if and only if } \vdash A \text{ in } G. \]

Proof.
A is valid iff \( \neg A \) is unsatisfiable iff there is a closed semantic tableau for \( \neg A \) iff there is a proof of \( A \) in \( G \).
Exercises

Prove the following in $\mathcal{G}$:

- $\vdash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$
- $\vdash (A \rightarrow B) \rightarrow ((\neg A \rightarrow B) \rightarrow B)$
- $\vdash ((A \rightarrow B) \rightarrow A) \rightarrow A$